Convex hull violation by superpositions of multifractals

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We elucidate the emergence of first- and second-order phase transitions in superpositions of multifractals. Using the description from the generalized entropy point of view, we resolve an observed violation of the convex hull principle. The approach considerably simplifies an earlier discussion and covers new classes of systems. [S1063-651X(97)00806-4]

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As a consequence of the success of nonlinear dynamics, the theory of fractals [1–5] has attracted much interest. Singular measures of multifractal structure [6] were shown to characterize strange attractors [7], diffusion [8,9], and scattering processes [4,10], turbulence, fractal diffusion-limited aggregation [3], etc. This point of view was recently extended to cover more complex cases by considering so-called spatially extended systems [11]. In a generalized sense, the system investigated below belongs to this class. It is the purpose of this paper to provide an exemplary discussion of the superposition of multifractals, where the discussion is from the entropy point of view.

Often in experiments the full multifractal structure of the object cannot be accessed. What can be measured is a fractal structure, which then typically has a support-independent measure on it ($\exists c \in \mathbb{R}: p_j = c\, l_j, \forall j$, see below). This measure can be the product of a partial overlap due to a projection from higher-dimensional spaces, or the measure arises quite naturally as the sum of different probability measures attached to the same Cantor structure. For this whole class of problems, only a few theoretical results have been obtained [12].

As a typical example for a partial overlap we mention the superposition of scattering trajectories on the surface of the scatterers in the Lorentz-gas problem or the coxistence of almost identical fractal attractors that can be observed in systems of dimension $d>3$. A sufficiently general model for different probability measures attached to a Cantor structure is provided by a neuron with two inputs. This neuron learns two patterns η_1, η_2, which are sent from a source uncorrelatedly and stochastically [13], where a binomial distribution with probabilities $p_1(1), p_2(1)$ for sending pattern 1 and 2, respectively, is assumed. At sufficiently high learning rates, upon using a gradient descent learning rule, a chaotic trajectory characterizes the state of the neuron. Suppose now that, after a number of steps of learning due to patterns sent by the first source, the neuron switches listening to a second source sending the same patterns, but with different probabilities $p_1(2), p_2(2)$. After another number of steps, again the neuron listens to the first source, and so on. In the associated phase space, this fact is hardly distinguishable from the case where only one source is present since both cases generate the *same* topological attractor. This situation is related to the model of *parallel iteration* [14], where distinct maps are iterated on the same topological attractor according to distinct probability distributions. (If we stick longer than the embedding time to one map, then the product of the systems is *not* observed since the trajectories determined by one map typically evade the neighborhood of trajectories determined by the other map much too fast.)

In Ref. [12], the superposition of two binary multifractals was considered and for this system the behavior of the $f(\alpha)$ function [3,5–7] was discussed. It was shown that (in addition to first-order phase transitions that can be expected to appear in such situations) transitions of second order emerge under specific conditions. The derived conditions, however, could not be formulated in simple terms and the presented proof was somewhat involved. In this paper it is shown how the description from the entropy point of view gives much simpler insight into this phenomenon. Furthermore, the findings of a previous work [12] are extended by working out similar properties of systems based on more complicated grammars and more than two sources. In addition, the discussion refers to a wider context because the generalized entropy point of view comprises all kinds of specific entropy properties and thus goes beyond the discussion of $f(\alpha)$.

For the model, suppose that the sources are complete, self-similar $\widetilde{M}$-scale multifractals. Each of these sources is then determined by the probabilities p_i, $i \in \{1,2,\ldots,\widetilde{M}\}$, and the associated length scales l_i [1–3]. The hierarchic structure of this multiplicative process is captured in the generalized partition sum [4–7,15]

$$Z(q,\beta,N) = \sum_j p_j^q l_j^\beta, \tag{1}$$

where N denotes the level of the construction hierarchy. Here $j \in \widetilde{M}^N$ is the *symbolic* address of an allowed contribution to the hierarchic process (for *multiplicative measures* we have $p_j = p_1 p_1 p_2 \cdots$ if $j = \{1,1,2,\ldots\}$ and allowed means that $p_j \neq 0$). The probabilities are normalized: $\Sigma_i p_i = 1$. In order to label M independent sources we use the index k.

Accordingly, the system is characterized by the set of quantities $p_i(k), l_i(k)$, $i=1,\ldots,\widetilde{M}$ and $k=1,\ldots,M$. In the following we consider the case of the superposition μ of M multifractal measures, i.e., $\mu=\Sigma_k\mu(k)$, where $k=1,\ldots,M$. A simple situation is obtained if the supports of the k measures are the same and the measures are multiplicative. This implies that $l_i(k)=l_i$ for all $i=1,\ldots,\widetilde{M}$ and $k=1,\ldots,M$. For *full binary* (i.e., $\widetilde{M}=2$) grammar the partition sum of the superposition has the form

$$Z(q,\beta,N)=\sum_{j=0}^{N}\binom{N}{j}\times\left(\sum_{k=1}^{M}\pi(k)p_1(k)^j p_2(k)^{(N-j)}\right)^q (l_1^j l_2^{(N-j)})^\beta. \tag{2}$$

In this formula the parameters $\pi(k)$, $k=1,2$, are the weights of the contributions of the involved multifractals $[\Sigma_k\pi(k)=1]$. For $\widetilde{M}>2$, the binomial coefficient has to be replaced by more involved expressions; nonfull grammars may sometimes be reduced to full grammars of different type [16]. Letting $\xi=j/N$, in order to evaluate the partition sum, Z is written as an integral [12]

$$Z(q,\beta,N)\sim\int_0^1 e^{-Ng(\xi,q,\beta)}d\xi \tag{3}$$

with, specifying for the simplest case $M=2$ and $\widetilde{M}=2$,

$$\begin{aligned}g(\xi,q,\beta)=&\,\xi\ln(\xi)+(1-\xi)\ln(1-\xi)+(\xi\{-q\ln[p_1(1)]\\&-\beta\ln(l_1)\}+(1-\xi)\{-q\ln[p_2(1)]\\&-\beta\ln(l_2)\})\theta(\xi-\xi_0)+(\xi\{-q\ln[p_1(2)]\\&-\beta\ln(l_1)\}+(1-\xi)\{-q\ln[p_2(2)]\\&-\beta\ln(l_2)\})\theta(\xi_0-\xi),\end{aligned} \tag{4}$$

where $\theta(x)$ denotes the step function. The special value ξ_0 is characterized by the equality of the two contributions; it has the value

$$\xi_0=\{1+\ln[p_1(1)/p_1(2)]/\ln[p_2(1)/p_2(2)]\}^{(-1)}. \tag{5}$$

Fractals reveal a deep connection to more traditional fields of physics due to a (formal) equivalence of multifractals with spin systems [17]. In analogy to the latter, thermodynamic quantities are defined, and nonanalytic behavior of these functions can be interpreted as phase transition phenomena [18]. Using this language, $F(q,\beta)=\lim_{N\to\infty}\ln[Z(q,\beta,N)]/N$ is the generalized free energy [7] and we have $F(q,\beta)=-g(\xi(q,\beta),q,\beta)$, where $\xi(q,\beta)$ determines the maximum of the integrand in Eq. (3) by minimizing the function g, for given q and β. Here we wish to elucidate the appearance of these phase transition phenomena from the generalized entropy point of view, which gives a straightforward, unified, and also easier access to the observed phenomena. The generalized entropy function $S(\alpha,\varepsilon)$ [7,14] arises as the Legendre transform of $F(q,\beta)$ as

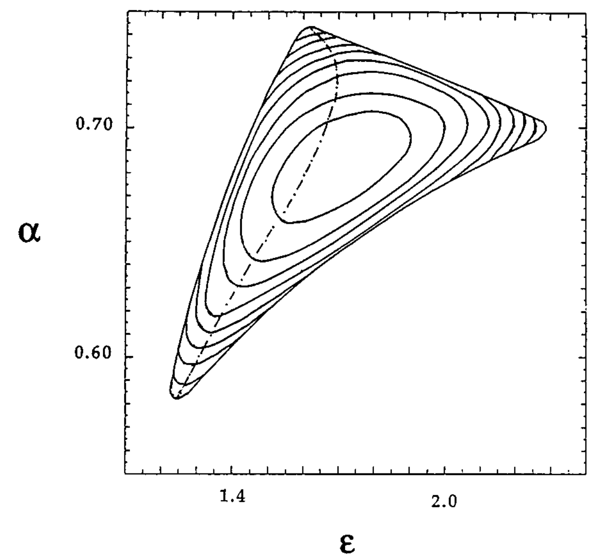

FIG. 1. Generalized entropy function for a ternary system. Contour plot, where the lines increase in steps of 0.1. The corner points characterize the scaling properties of the three symbols (‘‘pure states’’). The trace of the specific entropy function $f(\alpha)$ is indicated by a dashed line.

$$S(\alpha,\varepsilon)=F(q,\beta)+q\alpha\varepsilon+\beta\varepsilon. \tag{6}$$

Here $\varepsilon=-\partial F(q,\beta)/\partial\beta$ is the local scaling rate of the support and the ‘‘local’’ dimension arises as $\alpha=-[\partial F(q,\beta)/\partial q]/\varepsilon$. Note that it may be preferable to change the signs of ε in order to express a different interpretation of the length scales.

For the discussion of the generalized entropy spectrum we first recall the general case without superposition (or a superposition of two identical systems). Such systems do not show phase transition effects. For a system of ternary grammar, a typical entropy function as shown in Fig. 1 is obtained. Note the line indicating the trace along which the function $f(\alpha)$ is evaluated.

How can the trace be determined along which $f(\alpha)$ is evaluated? It is a simple consequence of the definition of $f(\alpha):=[S(\alpha,\varepsilon)/\varepsilon]|_{\beta(q):F(q,\beta(q))=0}$ that the trace can be thought to emerge from a simple experiment (cf. Fig. 1): Hold a ruler parallel to the ε axis, while the left end of the ruler is at the origin of the coordinate system. Move the ruler now towards higher α values, with one end still tied to the $(\varepsilon=0,\alpha)$ axis, while maintaining parallelism with respect to the ε axis. The tangential points of the ruler with the entropy surface then yield the line along which $f(\alpha)$ is evaluated. By a projection of this line the *trace* of $f(\alpha)$ in the (ε,α) plane is obtained.

The entropy function of a single binary system degenerates into a sheet with one-dimensional support due to the fact that the function $(\beta,q)\to(\varepsilon,\alpha)$ is *not injective* [19] (in the presence of phase transitions, a two-dimensional support can be obtained.) If we consider the superposition of two *binary systems*, we start off with sheets in the scaling plane (ε,α). It can easily be proved that these sheets have to intersect and that there are only four cases: the two cases shown in Fig. 2 and two cases obtained from their reflection about a line

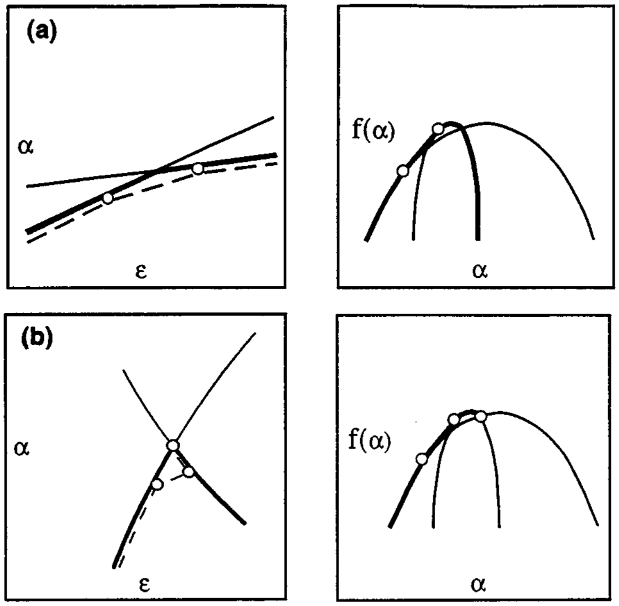

FIG. 2. Support obtained for a system of two binary sources (light lines) with relevant branches (heavy lines) in the (ε,α) plane. The $f(\alpha)$ spectrum is evaluated along the dashed lines; dots indicate the phase transition points. The graph of $f(\alpha)$ itself is shown by heavy lines; light lines indicate the $f(\alpha)$ curves associated with the two systems before superposition. (a) Both individual entropy sheets are ascending and (b) one is ascending, the other is descending. Note that the $f(\alpha)$ curve of the superimposed system is *not obtained* from the convex hull of the contributions.

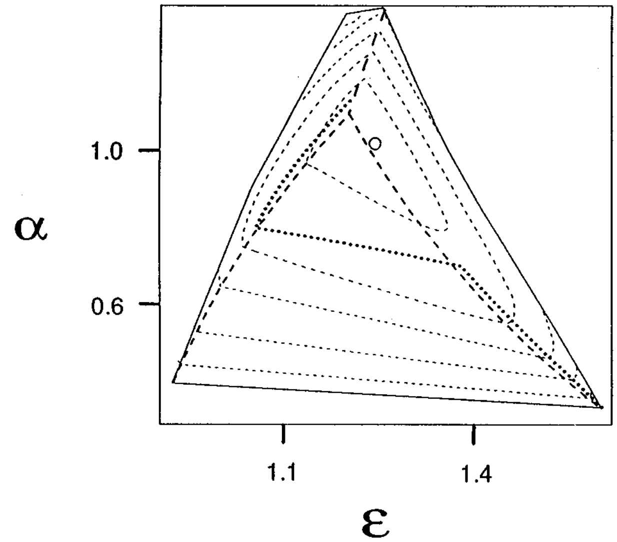

FIG. 3. Generalized entropy spectrum of a two-source three-pattern system ($M=2$, $\tilde{M}=3$). The trace of $f(\alpha)$ in the entropy function is indicated by dots. Heavy long-dashed line, second-order phase transition line, heavy short-dashed line, first-order phase transition line; light dashes, contour lines. In Figs. 2, the areas of first-order phase transitions are not drawn.

parallel to the α axis through their point of intersection. We will discuss only the situations shown in Fig. 2; the two remaining cases can be discussed analogously. The discussion is as follows. Each symbolic address j triggers one specific logarithmic length scale ε. However, each of the length scales is associated with two different measure exponents α_k, $k+1,\ldots,M=2$. The measure exponents correspond to probabilities, which themselves correspond to the measure on the Cantor piece labeled by j. Note now that in the asymptotic limit $N\rightarrow\infty$ the largest probability dominates and is the only one that matters (in our setting, $\Sigma p(j)_k^N$ and not $[\Sigma p(j)_k]^N$ is relevant for the measure). The largest $p(j)_k$ corresponds to the *smallest* $\alpha(j)_k$. Therefore, only the smallest α will survive. As a consequence, the upper wings are cut off for the entropy function of the superimposed system.

If now for the superimposed system q is monitored from ∞ to $-\infty$, in Fig. 2 the ruler starts moving up the branch that provides the lowest α value. At $(q=1,\beta=0)$, invariably a phase transition appears. This is because at this parameter set the free energies of both branches are zero. On the two branches, the free energies grow at different rates, which yields a first-order transition. The intersection point of the two branches is characterized by the equality of the free energies, the length, and the measure exponents. The point therefore corresponds to the second intersection of the $f(\alpha)$ curves. In Fig. 2(a), where both individual entropy curves are ascending, for $q\rightarrow-\infty$ $f(\alpha)$ follows the more narrow individual curve until zero entropy is reached. In Fig. 2(b), the $f(\alpha)$ curve stops at the intersection point. Because of the above characterization of the intersection point, this point itself can be understood as a phase of its own in the free-energy picture. Entering into this phase can therefore be interpreted as a second-order phase transition. In both cases, the resulting entropy function is *not* obtained from the convex hull of the contributing individual $f(\alpha)$ functions.

We put our work in a wider perspective by considering in addition to $f(\alpha)$ other specific entropy functions [20], which all arise by restriction of $S(\varepsilon,\alpha)$ according to some conditions [e.g., the restriction to $\beta=0$ yields the Legendre transform $g(\Lambda)$ of the Rényi entropies $K(q)$]. Most of these specific entropy functions also undergo a first-order transition. Actually, the whole area between the two branches is an area of first-order transitions that has to be added to the traces shown in Fig. 2. We see this more clearly if we consider the superposition of $M=2$ ternary systems. For one specific parameter set the result is shown in Fig. 3. As the most noticeable change in comparison with the superposition of two binary systems we note that the intersection point changed into an intersection line. Entering this line again indicates a second-order phase transition (compare with a corresponding result obtained for a related system in [21]). The numerical result shown in Fig. 3 is characteristic in the sense that this choice of parameters yields in a sense a minimal change of the picture if more than $M=2$ systems are superimposed. Increasing the number of contributing ternary maps with randomly chosen sets of probabilities yields asymptotically the entropy function as shown in Fig. 4. The convergence towards this asymptotic function is very fast (the form is established already for $M\sim 25$). When such ''universal'' entropy functions of different grammatical types are compared, an astonishing property can be detected [14]: Universal entropy functions of simpler grammatical type are identical to a part of the universal entropy function of higher grammatical type.

Nonhyperbolicities and superpositions are the main

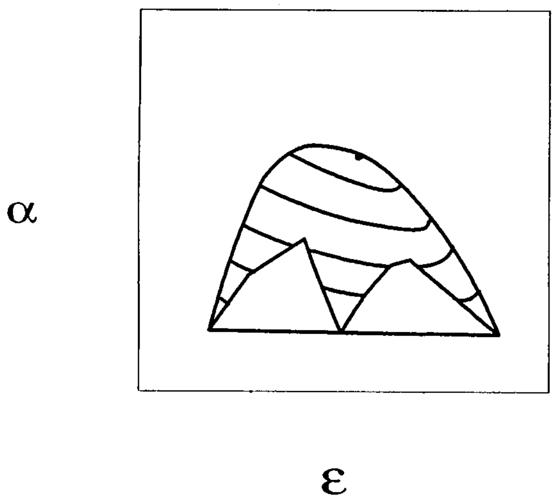

FIG. 4. Asymptotic entropy function, obtained for $N=2000$ ternary systems. The ''batlike'' upper part of the entropy sheet is filled up with second-order phase transition lines. (Contour lines are of distance 0.2.)

mechanisms for generating phase transitions in the often measured spectra of scaling indices. We would therefore like to put the present results into this broader context by pointing out some relations with earlier works that concentrated on phase transition effects generated by nonhyperbolicities. The above-discussed models can be seen in connection with models of multifractals of $\widetilde{M}=2$ and of $\widetilde{M}=3$ symbols, which are built from a nonhyperbolic measure on a hyperbolic support. There the influence of the phase transition results in a point of nonhyperbolicity far off from the hyperbolic sheet, whereas here the first order transition regime extends *between* the hyperbolic branches. In this comparison, Fig. 2 corresponds to the system treated in Ref. [19]; Fig. 3 then corresponds to the type of systems discussed in Ref. [20]. The latter models then were used in order to show that, upon suitably modifying the system parameters length and probability scales, phase transitions generated by nonhyperbolicities can be moved from one specific entropy spectrum into another specific entropy spectrum [in Ref. [20] from $f(\alpha)$ into $g(\Lambda)$].

In conclusion, the generalized entropy representation provides an excellent tool for the understanding of phase transition effects in multifractals. By using this tool we were able to explain previous findings in a more natural way. We indicated how our approach extends to the discussion of other specific entropy functions and we presented results for classes of multifractals that, to our knowledge, have not been treated before.

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